Function Theory for Laplace and Dirac-Hodge Operators in Hyperbolic Space

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Abstract

We develop basic properties of solutions to the Dirac-Hodge and Laplace equations in upper half space endowed with the hyperbolic metric. Solutions to the Dirac-Hodge equation are called hypermonogenic functions while solutions to this version of Laplace's equation are called hyperbolic harmonic functions. We introduce a Borel-Pompeiu formula for C^1 functions and a Green's formula for hyperbolic harmonic functions. Using a Cauchy Integral formula we are able to introduce Hardy spaces of solutions to the Dirac-Hodge equation. We also provide new arguments describing the conformal covariance of hypermonogenic functions and invariance of hyperbolic harmonic functions. We introduce intertwining operators for the Dirac-Hodge operator and hyperbolic Laplacian.

Keywords Clifford analysis, hypermonogenic functions, quasi-Cauchy's integral formula, Green's formula, Dirac-Hodge equation, hyperbolic harmonic functions

1 Introduction

Function theory for Dirac operators on manifolds have been developed in [4, 7, 27]. For particular types of manifolds this function theory has been developed in detail in [22, 19, 20, 21, 29, 31] and elsewhere. In this paper we develop a detailed function theory associated to the Dirac-Hodge operator and Laplacian in upper

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half space, for $n > 2$, endowed with the hyperbolic metric. Analysis of these operators have been developed over many years by many authors, particularly with respect to links to the Weinstein equation and its links to differential geometry and elasticity. See for instance [1, 3, 6, 8, 9, 11, 10, 12, 13, 15, 16, 17, 23, 24, 32].

Adapting the Cauchy Integral Formula for solutions to the Dirac-Hodge equation introduced in [9] we also introduce a Green's formula for hyperbolic harmonic functions, Borel-Pompeiu formulas and other representation formulas. In particular we are able to study basic properties of Hardy spaces and Plemelj projection operators for hyper-surfaces in upper half space. We also introduce a Poisson integral formula to this setting. This thereby sets up the tools necessary for studying boundary value problems in this context. We also describe the conformal invariance of the operators introduced here and describe intertwining operators for these operators under actions of Möbius transformations preserving upper half space.

2 Preliminaries

Here we will consider upper half space R^{n+} endowed with the hyperbolic metric $ds^2 = \frac{dx_1^2+\ldots+dx_n^2}{x_n^2}$. With respect to this metric one may consider the adjoint δ to the de Rham exterior derivative d. Namely $\delta = \star d \star$, where $\star$ is the Hodge star map acting on sections in the alternating bundle over $R^{n,+}$. The *Dirac-Hodge operator* is the differential operator $d + \delta$ acting on differentiable sections on the alternating algebra $\Lambda(R^{n,+})$. The square of $d + \delta$ is the Laplacian $d\delta + \delta d$ with respect to the hyperbolic or Poincare metric. To better understand the Dirac-Hodge operator let us first follow [24] and note that as an vector space the alternating or exterior algebra $\Lambda(R^n)$ is isomorphic to the Clifford algebra Cl_n generated from R^n with *negative* definite inner product. Namely let us consider R^n with orthogonal basis $e_1, \ldots, e_n$. Then Cl_n has as its basis

$$1, e_1, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_1 \ldots, e_n$$

and

$$e_1e_j + e_je_i = -2\delta_{ij}.$$

Hence an arbitrary element of the basis may be written as $e_A = e_{\alpha_1} \ldots e_{\alpha_h}$, here $A = \{\alpha_1, \ldots, \alpha_h\} \subseteq \{1, \ldots, n\}$ and $1 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_h \leq n$.

We may express the Clifford algebra as

$$Cl_n = Cl_{n-1} + Cl_{n-1}e_n,$$

where Cl_{n-1} is the Clifford algebra generated from R^{n-1} with orthonormal basis $e_1, \ldots, e_{n-1}$. So if $A \in Cl_n$ there are unique elements B and $C \in Cl_{n-1}$ such that $A = B + Ce_n$. This gives rise to a pair of projection maps

$$P : Cl_n \rightarrow Cl_{n-1} : P(A) = B$$

and

$$Q : Cl_n \rightarrow Cl_{n-1} : Q(A) = C.$$

We will denote $-e_nQ(A)e_n, \in Cl_{n-1}$, by $Q'(A)$.

The Dirac-Hodge operator, $d + \delta$ now retranslates in Clifford algebra notation as $D + \frac{n-2}{x_n}Q'$, where $D = \Sigma_{j=1}^n e_j \frac{\partial}{\partial x_j}$ is the euclidean Dirac operator. So the Dirac-Hodge equation is

$$Df + \frac{n-2}{x_n}Q'(f) = 0$$

where $f : U \rightarrow Cl_n$ is a differentiable function and U is a domain in $R^{n+} = \{x = x_1e_1 + \ldots + x_ne_n : x_n > 0\}$. See [24] for more details. We shall abbreviate the Dirac-Hodge equation to $Mf = 0$. It may readily be determined that:

Proposition 1 *Suppose that U is a domain in upper half space then the space of all solutions to the Dirac-Hodge equation $\{f(x) : x \in U$ and $Mf = 0\}$ is a right module with respect to the algebra Cl_{n-1}.*

Note, [11], that if U is a domain in upper half space and $h : U \rightarrow Cl_n$ is a C^2 function then

$$-M^2h = \triangle P(h) - \frac{n-2}{x_n}\frac{\partial P(h)}{\partial x_n} + \left(\triangle Q(h) - \frac{n-2}{x_n}\frac{\partial Q(h)}{\partial x_n} + \frac{n-2}{x_n^2}Q(h)\right)e_n$$

where $\triangle$ is the euclidean Laplacian.

In [1] it is noted for any real valued function $u(x)$ defined on the domain U then

$$\triangle u - \frac{n-2}{x_n}\frac{\partial u}{\partial x_n}$$

is the Laplace formula for upper half space with respect to the hyperbolic metric. We will denote this Laplacian by $\triangle_{R^{n,+}}$. We will call a Cl_{n-1} valued solution to the equation

$$\triangle h - \frac{n-2}{x_n}\frac{\partial u}{\partial x_n} = 0$$

a *hyperbolic harmonic function.* It follows that if f is hypermonogenic and C^2 then $P(f)$ is hyperbolic harmonic. Furthermore we shall denote the operator

$$\triangle - \frac{n-2}{x_n}\frac{\partial}{\partial x_n} + \frac{n-2}{x_n^2}$$

by $\triangle'_{R^{n,+}}$. The equations $\triangle_{R^{n,+}}u = 0$ and $\triangle'_{R^{n,+}}u = 0$ are both examples of the *Weinstein equation.* See for instance [3, 23, 32] for details.

Returning to the Clifford algebra, we will need the anti-automorphism

$$\sim : Cl_n \rightarrow Cl_n :\sim e_{j_1} \ldots e_{j_r} = e_{j_r} \ldots e_{j_1}.$$

One usually writes $\tilde{A}$ for $\sim A$. Also for $A, B \in Cl_n$ one has, [28], $\widetilde{AB} = \tilde{B}\tilde{A}$. So if $f : U \to Cl_n$ satisfies $Mf = 0$ then $\tilde{f}$ satisfies $fM = 0$ where $fM = \Sigma_{j=1}^n \frac{\partial f}{\partial x_j} e_j + \frac{n-2}{x_n} Q'(f)$.

Following [2, 30] one may express any Möbius transformation, $\phi(x)$, over $R^n \cup \{\infty\}$ as $(ax+b)(cx+d)^{-1}$ where a, b, c and d are products of vectors from R^n and $\tilde{a}c$, $\tilde{c}d$, $\tilde{d}b$ and $\tilde{b}a \in R^n$. Moreover a, b, c and d are all products of vectors from R^n and we may assume that $\tilde{a}d - \tilde{b}c = \pm 1$. This gives rise to a covering group, $V(n)$, of the group of Möbius transformations over $R^n \cup \{\infty\}$. We will be interested in the subgroup $V(n-1)$ that acts on R^{n-1}. The group $V(n)$ is often called the Vahlen group. Following [5] for any four vectors w_1, w_2, w_3 and $w_4 \in R^n$ we define their *cross ratio,*

$$[w_1, w_2, w_3, w_4], \quad \text{to be} \quad (w_1 - w_4)^{-1}(w_1 - w_3)(w_2 - w_3)^{-1}(w_2 - w_4).$$

Taking $A = a_0 + \ldots + a_{1\ldots n} e_1 \ldots e_n \in Cl_n$ we define the norm of A to be, as usual, $\|A\| = (a_0^2 + \ldots + a_{1\ldots n}^2)^{\frac{1}{2}}$. Using the conjugation antiautomorphism $- : Cl_n \to Cl_n : \underline{-}(e_{j_1} \ldots \underline{e_{j_r}}) = (-1)^r e_{j_r} \ldots e_{j_1}$ it may be seen that $\|A\|^2$ is the real part of $A\overline{A}$, where $\overline{A}$ denotes the conjugate of A. It may be seen that if $A = \underline{x}_1 \ldots \underline{x}_k$ and each $\underline{x}_j \in R^n$ for $1 \leq l \leq k$ then $\|A\|^2 = \|\underline{x}_1\|^2 \ldots \|\underline{x}_k\|^2$. Each Möbius transformation $\psi(x)$, $= (ax+b)(cx+d)^{-1}$ can be expressed as $ac^{-1} \pm (cx\tilde{c} + d\tilde{c})^{-1}$ whenever $c \neq 0$ and $\psi(x) = \alpha a x \tilde{a} + bd^{-1}$ for some $\alpha \in R$ whenever $c = 0$. Consequently:

Lemma 1 *For each Möbius transformation ψ*

$$\|[w_1, w_2, w_3, w_4]\| = \|[\psi(w_1), \psi(w_2), \psi(w_3), \psi(w_4)]\|.$$

This invariance of the norm of the cross ratio is also noted in [1].

The *Cayley transformation* of upper half space $R^{n,+}$, $= \{x = x_1 e_1 + \ldots + x_n e_n : x_n > 0\}$ to the unit ball is given by

$$C(x) = (e_n x + 1)(x + e_n)^{-1} = e_n(x - e_n)(x + e_n)^{-1}.$$

This transformation maps e_n to the origin. If we wanted to adapt this transformation to a Cayley type Möbius transformation that maps upper half space to the unit ball and maps a point y in upper half space to the origin then one has the Möbius transformation

$$C(x, y) = e_n(x - y)(x - \hat{y})^{-1}$$

where $\hat{y} = y_1 e_1 + \ldots + y_{n-1} e_{n-1} - y_n e_n$. So $\hat{y}$ is the reflection of y about $R^{n-1} = span < e_1, \ldots, e_{n-1} >$. Note that

$$\|C(x, y)\| = \frac{\|x - y\|}{\|x - \hat{y}\|} = \|[x, \hat{x}, y, \hat{y}]\|^{\frac{1}{2}}.$$

Consequently we have the following simple but important result.

Lemma 2 *Suppose that $\psi \in V(n-1)$. Then $\psi(\hat{y}) = \hat{\psi}(y)$ and*

$$\|C(x,y)\| = \|C(\psi(x), \psi(y))\|.$$

As a consequence of this lemma one also has:

Proposition 2 *Suppose $f : [0, \infty) \to Cl_{n-1}$ is an L^1 function and $\psi \in V(n-1)$ then*

$$F(x,y) = \int_0^{\frac{\|x-y\|}{\|x-\hat{y}\|}} f(r)dr$$

is a well defined function on $R^{n,+} \times R^{n,+}$ and $F(\psi(x), \psi(y)) = F(x,y)$.

3 Some Cauchy and Green's Integral Formulas

Following [1] let us first note that the hyperbolic Laplace equation on the unit ball in R^n is

$$\triangle_{B(0,1)} u = \triangle u + \frac{2(n-2)r}{1-r^2}\frac{\partial u}{\partial r} = 0.$$

Again following [1] suppose now that $u(x)$ is a hyperbolic harmonic function depending only on $r = |x|$. First one obtains

$$\frac{\partial u}{\partial x_i} = u'(r)\frac{x_i}{r}$$

and

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = u''(r)\frac{x_i x_j}{r^2} + u'(r)\left(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}\right).$$

Thus

$$\triangle u = u'' + (n-1)\frac{u'}{r}.$$

As $u(x)$ is a hyperbolic harmonic function $u(r)$ will satisfy

$$u'' + (n-1)\frac{u'}{r} + \frac{2(n-2)}{1-r^2} r u' = 0.$$

If $u' \neq 0$ this can be written as

$$\frac{u''}{u'} + \frac{n-1}{r} + \frac{2(n-2)r}{1-r^2} = 0.$$

or

$$\frac{d}{dr}[\log u' + (n-1)\log r - (n-2)\log(1-r^2)] = 0$$

from which we conclude that

$$u'(r)\frac{r^{n-1}}{(1-r^2)^{n-2}} = const.$$

This leads to the general solution

$$u(r) = a\int_1^r \frac{(1-t^2)^{n-2}}{t^{n-1}}dt + b.$$

We see at once that no solution can stay finite for $r = 0$. As a normalized solution we introduce

$$g(r) = \int_r^1 \frac{(1-t^2)^{n-2}}{t^{n-1}}dt.$$

From Proposition 1 it now follows that the real valued function

$$G(x,y) = \int_{\frac{\|x-y\|}{\|x-\hat{y}\|}}^1 \frac{(1-t^2)^{n-2}}{t^{n-1}}dt$$

is a hyperbolic harmonic function. As $G(x,y)$ is real valued then trivially $Q(G(x,y)) = 0$. Consequently $MG(x,y) = DG(x,y)$. Therefore, [24], the function $p(x,y) = DG(x,y)$ is a vector valued hypermonogenic function. Here M and D are acting with respect to the x variable. Following [9, 12] it may be noted that

$$\begin{aligned}
DG(x,y) &= \frac{(1-s^2)^{n-2}}{s^{n-1}}\Big|_{\frac{\|x-y\|}{\|x-\hat{y}\|}}^1 D\frac{\|x-y\|}{\|x-\hat{y}\|} \\
&= \frac{(4x_ny_n)^{n-2}}{\|x-y\|^{n-1}\|x-\hat{y}\|^{n-3}}\Sigma_{j=1}^n e_j\frac{\partial}{\partial x_j}\frac{\|x-y\|}{\|x-\hat{y}\|} \\
&= \frac{(4x_ny_n)^{n-2}}{\|x-y\|^{n-1}\|x-\hat{y}\|^{n-3}}\left(\frac{x-y}{\|x-y\|\|x-\hat{y}\|} - \frac{(x-\hat{y})\|x-y\|}{\|x-\hat{y}\|^3}\right) \\
&= (4x_ny_n)^{n-2}\left(\frac{(x-y)^{-1}}{\|x-y\|^{n-2}\|x-\hat{y}\|^{n-2}} - \frac{(x-\hat{y})^{-1}}{\|x-y\|^{n-2}\|x-\hat{y}\|^{n-2}}\right) \\
&= x_n^{n-2}y_n^{n-1}\left(\frac{(x-y)}{\|x-y\|^n}e_n\frac{(x-\hat{y})}{\|x-\hat{y}\|^n}\right).
\end{aligned}$$

Suppose now that U is a domain in upper half space and for two C^1 functions f and g defined on U and taking values in Cl_n we consider the integral $\int_S g(x)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)$, where S is a compact smooth hypersurface lying in U, $n(x)$ is the outer unit normal vector to S at x and σ is the Lebesgue surface

measure of S. On assuming that S is the boundary of a bounded subdomain V of U then on applying Stokes' Theorem we obtain

$$\int_S g(x)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)$$
$$= \int_V ((g(x)D)\frac{1}{x_n^{n-2}}f(x) + g(x)\frac{1}{x_n^{n-2}}Df(x) - g(x)\frac{(n-2)}{x_n^{n-1}}e_n f(x)dx^n.$$

It follows that:

Lemma 3 *[12] Suppose, f, g, U, S and V are as in the previous paragraph. Then*

$$P\left(\int_S g(x)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)\right) = P\left(\int_V (g(x)M))f(x) + g(x)(Mf(x))\frac{dx^n}{x_n^{n-2}}\right).$$

Consequently if $Mf = 0$ and $gM = 0$ we have the version of Cauchy's Theorem established in [12]. Namely $\int_S g(x)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x) = 0$. It may now be determined that for each $y \in V$

$$P(f(y)) = \frac{2^{n-2}}{\omega_n}P\left(\int_S p(x,y)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)\right).$$

This is the Cauchy integral formula arising in [12]. It is an easy consequence of this integral formula and the previous lemma to obtain:

Theorem 1 (Borel Pompeiu Theorem) *Suppose that $f : U \to Cl_n$ is a C^1 function and that U is a bounded open subset of upper half space with C^1 compact boundary lying in upper half space. Suppose also that f has a continuous extension to the boundary of U. Then for each $y \in U$*

$$P(f(y)) = \frac{1}{\omega_n}P\left(\int_{\partial U} p(x,y)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x) + \int_U p(x,y)(Mf(x))\frac{dx^n}{x_n^{n-2}}\right).$$

Clearly if $f(y) \in Cl_{n-1}$ then this integral would give $f(y)$.
It follows from this integral formula that if ϕ is a C^∞ function with values in Cl_{n-1} and with compact support on upper half space then

$$\phi(y) = \frac{1}{\omega_n}\int_{R^{n,+}} p(x,y)(M\phi(x))\frac{dx^n}{x_n^{n-2}}$$

for each $y \in R^{n,+}$. We also have as a consequence of Lemma 3 the following version of Green's Representation Formula for hyperbolic harmonic functions.

Theorem 2 (Green's Formula)*Suppose that U is a domain in upper half space and that $h : U \to Cl_{n-1}$ is a hyperbolic harmonic function. Then for S a piecewise C^1, compact surface lying in U and bounding a bounded subdomain V of U*

$$h(y) = \frac{1}{\omega_n} P\left(\int_S G(x,y)\frac{n(x)}{x_n^{n-2}}\Big(Mh(x)) - p(x,y)\frac{n(x)}{x_n^{n-2}}h(x)\Big)d\sigma(x)\right)$$

for each $y \in V$.

Stokes' Theorem also gives that if ϕ is Cl_{n-1} valued, C^∞, is defined on upper half space and has compact support then for each $y \in R^{n,+}$

$$\phi(y) = \frac{1}{\omega_n}\int_{R^{n,+}} G(x,y)\big(\triangle_{R^{n,+}}\phi(x)\big)\frac{dx^n}{x_n^{n-2}}. \tag{1}$$

Now let us consider $D_y G(x,y)$ where $D_y = \Sigma_{j=1}^n e_j \frac{\partial}{\partial y_j}$. As $\|x-\hat{y}\| = \|y-\hat{x}\|$ then $G(x,y)$ is hyperbolic harmonic in both the variables x and y, and

$$D_y G(x,y) = D_y \int_{\frac{\|y-x\|}{\|y-\hat{x}\|}}^1 \frac{(1-s^2)^{n-2}}{s^{n-1}} ds$$

$$= 4x_n^{n-2}y_n^{n-2}\left(\frac{(y-x)^{-1}}{\|x-y\|^{n-2}\|y-\hat{x}\|^{n-2}} - \frac{(y-\hat{x})^{-1}}{\|x-y\|^{n-2}\|x-y\|^{n-2}}\right) = h(x,y)$$

is hypermonogenic in the variable y.

Let M_y denote the Dirac-Hodge operator with respect to the variable y and let $\triangle_{R^{n,+},y}$ denote the hyperbolic Laplacian with respect to the variable y.

Theorem 3 *Suppose that ψ is a Cl_{n-1} valued, C^∞ function with compact support on upper half space. Then*

$$P\left(M_y\left(\frac{1}{\omega_n}\int_{R^{n,+}} h(x,y)\psi(x)\frac{dx^n}{x_n^{n-2}}\right)\right) = \psi(y).$$

Now consider

$$\triangle_{R^{n,+},y}\left(\frac{1}{\omega_n}\int_{R^{n,+}} G(x,y)\psi(x)\frac{dx^n}{x_n^{n-2}}\right).$$

This is equal to

$$\frac{1}{\omega_n}P\left(M_y\left(D\int_{R^{n,+}} G(x,y)\psi(x)\frac{dx^n}{x_n^{n-2}}\right)\right),$$

which in turn is equal to

$$\frac{1}{\omega_n}P\left(M_y\left(\int_{R^{n,+}} h(x,y)\psi(x)\frac{dx_n}{x_n^{n-2}}\right)\right).$$

By Theorem 3 this evaluates to $\psi(y)$. So we have established:

Theorem 4 *Suppose ψ is as in Theorem 3 then*

$$\triangle_{R^{n,+},y}\left(\frac{1}{\omega_n}\int_{R^{n,+}} G(x,y)\psi(x)\frac{dx^n}{x_n^{n-2}}\right) = \psi(y).$$

In [9] the kernel

$$q(x,y) = DH(x,y) = \frac{1}{2(n-2)} D \frac{1}{\|x-y\|^{n-2}\|x-\hat{y}\|^{n-2}},$$

where

$$H(x,y) = \frac{1}{(n-2)\|x-y\|^{n-2}\|x-\hat{y}\|^{n-2}}$$

is introduced. In [9] it is shown that the kernel $q(x,y)$ is the Cauchy kernel for the Q part of a Cauchy Integral Formula for hypermonogenic functions. So from [9] we have

$$f(y) = P(f(y)) + Q(f(y))e_n =$$

$$\frac{2^{n-1}y_n^{n-2}}{\omega_n}\left(P\left(\int_{\partial U} r(x,y)\frac{n(x)}{x_n^{n-2}}f(x))d\sigma(x)\right) - Q\left(\int_{\partial U} q(x,y)n(x)f(x)d\sigma(x)\right)e_n\right)$$

where $r(x,y) = y_n^{-n+2}p(x,y)$.
Again as a consequence of Stokes' Theorem we have:

Theorem 5 *Suppose that ϕ be a Cl_n valued C^1 function defined on a bounded domain $U \subset R^{n,+}$, with piecewise smooth boundary, and ϕ has a continuous extension to the closure of U. Then for each $y \in U$*

$$Q(\phi(y)) = \frac{2^{n-2}y_n^{n-2}}{\omega_n} Q\left(\int_{\partial U} q(x,y)n(x)\phi(x)d\sigma(x) - \int_U q(x,y)(M\phi(x))dx^n\right).$$

It follows immediately that if ϕ has compact support then

$$Q(\phi(y)) = \frac{2^{n-2}y_n^{n-2}}{\omega_n} Q\left(\int_{R^{n,+}} q(x,y)\big(M\phi(x)\big)dx^n\right).$$

Furthermore it may readily be determined that:

Theorem 6 (Green's Formula:)*Suppose that $u: U \to Cl_{n-1}e_n$ is a solution of the equation $\triangle'_{R^{n,+}}u = 0$, and U is as in Theorem 5. Then for each $y \in U$ we have*

$$u(y) = \frac{2^{n-2}y_n^{n-2}}{\omega_n} Q\left(\int_{\partial U} H(x,y)n(x)\big(Mu(x)\big) - q(x,y)n(x)u(x)d\sigma(x)\right).$$

In particular if u is a real valued function satisfying $\triangle'_{R^{n,+}}u = 0$ then

$$u(y) = -e_n\frac{2^{n-2}y_n^{n-2}}{\omega_n}\int_{\partial U} H(x,y)n(x)M\big(e_n u(x)\big) - q(x,y)n(x)e_n u(x)d\sigma(x).$$

By similar arguments to those used before we also have:

Theorem 7 *Suppose the U is as in Theorem 5 and $u : U \to Cl_{n-1}e_n$ is a C^2 function then*

$$u(y) = \frac{y_n^{n-2}}{\omega_n} Q\left(\int_{\partial U} H(x,y)n(x)\big(Mu(x)\big) - q(x,y)n(x)u(x)d\sigma(x) - \int_U H(x,y)\big(\triangle'_{R^{n,+}}u(x)\big)dx^n\right).$$

Consequently if u has compact support then

$$u(y) = \frac{y_n^{n-2}}{\omega_n} \int_{R^{n,+}} H(x,y)\big(\triangle'_{R^{n,+}}u(x)\big)dx^n.$$

In [23] within Lemma 2.1 it is shown that if $\phi(x)$ is a solution to

$$\triangle\phi(x) - \frac{n-2}{x_n}\frac{\partial\phi(x)}{\partial x_n} + \frac{n-2}{x_n^2}\phi(x) = 0$$

then $\theta(x) = x_n^{n-2}\phi(x)$ is a solution to the equation

$$\triangle\theta(x) - \frac{n-2}{x_n}\frac{\partial\theta(x)}{\partial x_n} + \frac{n-2}{x_n}\theta(x) = 0.$$

As $\|\hat{y} - x\| = \|y - \hat{x}\|$ it follows that $y^{n-2}H(x,y)$ is hyperbolic harmonic in the y variable. So by simple adaptations of standard arguments we also have

Proposition 3 *Suppose $u : R^{n,+} \to R$ is a C^2 function with compact support. Then*

$$u(y) = \triangle'_{R^{n,+}}\left(\frac{1}{\omega_n}y_n^{n-2}\int_{R^{n,+}} H(x,y)u(x)dx^n\right)$$

for each $y \in R^{n,+}$.

Now for any $A \in Cl_n$, $P(A) = \frac{1}{2}(A + \hat{A})$ where $\hat{A} = B - Ce_n$ with B and $C \in Cl_{n-1}$. Moreover, $Q(A) = \frac{-1}{2}(A - \hat{A})e_n$ and for any elements X and $Y \in Cl_n$ it is straightforward to determine that $\widehat{XY} = \hat{X}\hat{Y}$. Using these observations it is noted in [9] that the previous integral becomes

$$\frac{1}{\omega_n}2^{n-1}y_n^{n-2}\left(\int_{\partial U}\frac{1}{2}\Big(r(x,y)n(x)\frac{n(x)}{x_n^{n-2}}f(x) + \hat{r}(x,y)\frac{\hat{n}(x)}{x_n^{n-2}}\hat{f}(x)\Big)d\sigma(x) - \int_{\partial U}\frac{e_n}{2}\Big(q(x,y)n(x)f(x) - \hat{q}(x,y)\hat{n}(x)\hat{f}(x)\Big)d\sigma(x)\right).$$

In [9] it is shown that this expression simplifies to

$$f(y) = \frac{2^{n-1}y_n^{n-2}}{\omega_n}\left(\int_{\partial K}\frac{(x-y)^{-1}n(x)f(x)}{\|x-y\|^{n-2}\|x-\hat{y}\|^{n-2}}d\sigma(x) - \int_{\partial K}\frac{(\hat{x}-y)^{-1}\hat{n}(x)\hat{f}(x)}{\|x-y\|^{n-2}\|\hat{x}-y\|^{n-2}}d\sigma(x)\right).$$

If we write $E(x,y)$ for $\frac{(x-y)^{-1}}{\|x-y\|^{n-2}\|x-\hat{y}\|^{n-2}}$ and $F(x,y)$ for $\frac{(\hat{x}-y)^{-1}}{\|x-y\|^{n-2}\|\hat{x}-y\|^{n-2}}$ then this integral formula simplifies to

$$f(y) = \frac{2^{n-1}y_n^{n-2}}{\omega_n}\int_S \Big(E(x,y)n(x)f(x) - F(x,y)\hat{n}(x)\hat{f}(x)\Big)d\sigma(x).$$

4 Plemelj Projection Operators and Hardy Spaces of Hypermonogenic Functions

First let us note that as y_n tends to infinity then $y_n^{n-2}E(x,y)$ and $y_n^{n-2}F(x,y)$ both tend to zero for fixed x. Also as y_n tends to zero then both $y_n^{n-2}E(x,y)$ and $y_n^{n-2}F(x,y)$ tend to zero for fixed x.

Proposition 4 *Suppose that $C \in Cl_n$ is a constant and S is a compact, C^2 surface lying in upper half space. Suppose further that S is the boundary of a bounded domain U in $R^{n,+}$. If $y(t)$ is a C^1 path in U^+ with nontangential limit $y(1) = y \in S$ then*

$$\lim_{t\to 1}\frac{2^{n-2}y(t)_n^{n-2}}{\omega_n}\int_S \Big(E(x,y)n(x)C - F(x,y)\hat{n}(x)\hat{C}\Big)d\sigma(x) =$$

$$\frac{1}{2}C + \frac{2^{n-2}y_n^{n-2}}{\omega_n}PV\int_S \Big(E(x,y)n(x)C - F(x,y)\hat{n}(x)\hat{C}\Big)d\sigma(x).$$

Proof: Given that

$$\lim_{x\to y(1)}\frac{2^{n-2}y(1)_n^{n-2}}{\|\hat{x}-y(1)\|^{n-2}} = 1$$

then as S is compact it follows from the Mean Value Theorem that given $\epsilon > 0$ then for all $x \in S$ such that $\|x - y(1)\| < 1$ we have

$$\left|\frac{2^{n-2}y(1)_n^{n-2}}{\|\hat{x}-y(1)\|^{n-2}} - 1\right| < C'\|x-y(1)\|$$

and $C' \geq 0$. Let $S_\epsilon(y) = \{x \in S : \|x-y\| < \epsilon\}$. It follows that

$$\begin{aligned}
&\frac{2^{n-2}y(t)_n^{n-2}}{\omega_n}\int_{S_\epsilon(y)} E(x-y)n(x)Cd\sigma(x)\\
&\quad = \frac{1}{\omega_n}\int_S \frac{(x-y(t))}{\|x-y(t)\|^n}n(x)C\,d\sigma(x)\\
&\qquad + \frac{1}{\omega_n}\int_{S_\epsilon(y)}\left(\frac{2^{n-2}y(t)_n^{n-2}}{\|\hat{x}-y(t)\|^{n-2}} - 1\right)\frac{(x-y)}{\|x-y(t)\|^n}n(x)Cd\sigma(x).
\end{aligned}$$

It follows from the usual calculations, see [18], for Plemelj formulas in Clifford analysis that

$$\lim_{t\to 1}\frac{1}{\omega_n}\int_{S_\epsilon(y)}\frac{\big(x-y(t)\big)}{\|x-y(t)\|^n}n(x)Cd\sigma(x) = \frac{1}{2}C$$

and the other term tends to zero at t tends to 1. The result now follows. Q.E.D.
Although in the previous proposition we assumed that the surface is C^2 one can also prove this result for surfaces that are strongly Lipschitz. These are surfaces that are locally Lipschitz graphs and whose Lipschitz constants are uniformly bounded.
One also readily has the following important technical result.

Lemma 4 *For $x \in R^{n,+}$ and fixed $y \in R^{n,+}$ with $\|x-y\| > 2y_n$ then $\|E(x,y)\| < \frac{C}{\|x-y\|^{2n-2}}$ and $\|F(x,y)\| < \frac{C}{\|x-y\|^{2n-2}}$ for some $C \in R^+$.*

Using this lemma one can adapt arguments developed in [14, 26] and elsewhere to deduce:

Theorem 8 *Suppose that Σ is a Lipschitz graph lying in upper half space and the minimal distance between Σ and the boundary, R^{n-1}, of upper half space is greater than zero then the singular integral operator T_Σ defined by*

$$\frac{2^{n-2}}{\omega_n} PV \int_\Sigma y_n^{n-2} \left(E(x,y)n(x)\phi(x) - F(x,y)\hat{n}(x)\hat{\phi}(x) \right) d\sigma(x)$$

is L^p bounded for $1 < p < \infty$.

Clearly this result also holds if we replace the Lipschitz graph Σ by a compact, strongly Lipschitz surface S. In this case the operator T_Σ is replaced by its analogue T_S.
This result enables us to establish the analogues of Plemelj formulas in the present context.

Theorem 9 *Suppose that S is a compact, strongly Lipschitz surface lying in upper half space. Suppose also that S is the boundary of a bounded domain U^+ and an exterior domain $U^- \subset R^{N,+}$. Then for each function $\phi \in L^p(S)$ for $1 < p < \infty$ or a path $y_\pm(t) \in U^\pm$ with nontangential limit $y(1) = y \in S$ we have*

$$\lim_{t \to 1} \frac{2^{n-2} y_\pm(t)^{n-2}}{\omega_n} \int_S \left(E\big(x, y(t)\big)n(x)\phi(x) - F\big(x, y(t)\big)\hat{n}(x)\hat{\phi}(x) \right) d\sigma(x)$$

$$= \pm\frac{1}{2}\phi(y) + \frac{2^{n-2}}{\omega_n} PV \int_S y_n^{n-2} \left(E(x,y)n(x)\phi(x) - F(x,y)\hat{n}(x)\hat{\phi}(x) \right) d\sigma(x)$$

for almost all $y \in S$.

A minor adaptation of the proof of Theorem 17 in [9] tells us the following:

Theorem 10 *Suppose S is a Lipschitz surface lying in the closure of upper half space and $\phi \in L^p(S)$ for some $p \in (1, \infty)$ then the integral*

$$\frac{2^{n-2} y_n^{n-2}}{\omega_n} \int_S \left(E(x,y)n(x)\phi(x) - F(x,y)\hat{n}(x)\hat{\phi}(x) \right) d\sigma(x)$$

defines a left hypermonogenic function $f(y)$ on $R^{n,+} \backslash S$.

As $\lim_{y_n\to\infty} y_n^{n-2}E(x,y) = 0$ and $\lim_{y_n\to\infty} y_n^{n-2}F(x,y) = 0$ for each $x \in S$ and $\lim_{y_n\to 0} y_n^{n-2}E(x,y) = \lim_{y_n\to 0} y_n^{n-2}F(x,y) = 0$ for each $x \in S$ then $lim_{y_n\to\infty} f(y) = \lim_{y_n\to 0} f(y) = 0$. It now follows that the operators

$$\frac{1}{2}I \pm T_S : L^p(S) \to L^p(S)$$

are projection operators with images the Hardy spaces

$$H^p(U^\pm) = \{f : U^\pm \to Cl_n : f \text{ is left hypermonogenic and}\\ \text{nontangentially approaches some element in } L^p(S)\}.$$

Consequently

$$L^p(S) = H^p(U^+) \oplus H^p(U^-).$$

The operators $\frac{1}{2}I \pm T_S$ are generalizations of the Plemelj projection operators to the context of hypermonogenic functions. As in the euclidean case these operators are projection operators, or mutually annihilating idempotents. Let us denote the operator $\frac{1}{2}I + T_S$ by $\mathcal{H}_{\mathcal{S}}$. We may introduce the *Kerzman-Stein operator* $A_S = \mathcal{H}_{\mathcal{S}} - \mathcal{H}_{\mathcal{S}}^\star$, where $\mathcal{H}_{\mathcal{S}}^\star$ is the adjoint of $\mathcal{H}_{\mathcal{S}}$. In particular if $\phi \in L^2(S)$ then

$$A_S(\phi) = \frac{2^{n-2}y_n^{n-2}}{\omega_n}\left(\int_{\partial K}\big(E(x,y)n(x) - n(x)E(x,y)\big)\phi(x)\right.\\ \left. - \big(F(x,y)\hat{n}(x) + \hat{n}(x)F(x,y)\big)\hat{\phi}(x)d\sigma(x)\right).$$

Let us now turn to consider the case where $S = R^{n,+}$. We begin with:

Theorem 11 *Suppose $\phi \in L^p(R^{n-1})$ for some $p \in (1,\infty)$ then for $y(t) = y' + y_t e_n$, where $y' \in R^{n-1}$ and $y_n(t) > 0$,*

$$\lim_{t\to 0}\frac{2^{n-2}y_n(t)^{n-2}}{\omega_n}\left(\int_{R^{n-1}} E(x,y(t))e_n\phi(x) + F(x,y(t))e_n\hat{\phi}(x)dx^{n-1}\right) = P(\phi(y'))$$

almost everywhere.

Proof: Without loss of generality we may assume that $y' = \underline{0}$. Let us assume that ϕ is C^∞ and has compact support. Now for any $\epsilon > 0$

$$\lim_{t\to 0}\frac{2^{n-2}y_n(t)^{n-2}}{\omega_n}\left(\int_{R^{n-1}\setminus B(0,\epsilon)} E(x,y_ne_n(t))e_n\phi(x) + F(x,y_n(t)e_n)e_n\hat{\phi}(x)dx^{n-1}\right)$$

is equal to zero.

Further by arguments similar to those used to establish Proposition 4 we have:

$$\lim_{\epsilon\to 0, t\to 0}\frac{2^{n-2}y_n(t)^{n-2}}{\omega_n}\int_{B(0,\epsilon)}E(x,y_n(t)e_n)e_n\phi(x)dx^{n-1}=\frac{1}{2}\phi(0).$$

As $\hat{x}=x$ for each $x\in R^{n-1}$ then similarly

$$\lim_{\epsilon\to 0, t\to 0}\frac{2^{n-2}y_n(t)^{n-2}}{\omega_n}\int_{B(0,\epsilon)}F(x,y_n(t)e_n)e_n\hat{\phi}(x)dx^{n-1}=\frac{1}{2}\hat{\phi}(y').$$

A standard density argument now reveals the result for all $\phi\in L^p(R^{n-1})$. Q.E.D.

Theorem 11 tells us that in the special case where $S=R^{n-1}$ we may solve a Dirichlet problem for Cl_{n-1} valued L^p boundary data and for the Dirac-Hodge equation as opposed to the hyperbolic Laplace equation. This is in contrast to the euclidean analogues where one obtains a Plemelj formula for such data..

Suppose now that $\phi(x)$ is a real valued, L^p function defined on R^{n-1}, with $1<p<\infty$. Then on restricting to the real part of our previous integral we have the following Poisson integral

$$F(y)=\frac{2^{n-2}y_n^{n-1}}{\omega_n}\left(\int_{R^n}\frac{\phi(x)}{\|x-y\|^n\|x-\hat{y}\|^{n-2}}+\frac{\phi(x)}{\|x-y\|^{n-2}\|x-\hat{y}\|^n}d\sigma(x)\right),$$

which defines a hyperbolic harmonic function on upper half space with boundary value ϕ.

The material developed in this section enables one to tackle boundary values problems for the hyperbolic harmonic equation and for the equation $\triangle'_{R^{n,+}}u=0$. This includes problems like the Dirichlet and Neumann problems. One may adapt arguments given in [25, 26] to the context described here and solve such boundary value problems for hyperharmonic functions. This will be done elsewhere.

5 Representation Theorems

We begin with:

Theorem 12 (Borel-Pompeiu Formula) *Let $K\subset R^{n,+}$ be a bounded region with smooth boundary in $R^{n,+}$, Suppose also that $f:K\to Cl_n$ is a C^1 function on K with a continuous extension to the closure of K. Then for $y\in K$ we have*

$$f(y)=\frac{(2y_n)^{n-1}}{\omega_n}\int_{\partial K}P\Big(p(x,y)\frac{n(x)}{x_n^{n-2}}f(x)\Big)d\sigma(x)+\frac{1}{y_n}Q\big(q(x,y)n(x)f(x)\big)d\sigma(x)e_n$$
$$-\frac{(2y_n)^{n-1}}{\omega_n}\int_K\left[P\big(p(x,y)Mf\big)\frac{1}{x_n^{n-1}}+Q\big(q(x,y)Mf\big)\frac{e_n}{y_n}\right]dx^n$$

or

$$f(y)=\frac{(2y_n)^{n-2}}{\omega_n}\int_{\partial K}E(x,y)n(x)f(x)d\sigma(x)$$

$$-\int_{\partial K} F(x,y)\hat{n}(x)\hat{f}(x)d\sigma(x) - \frac{(2y_n)^{n-2}}{\omega_n}\int_K \left(E(x,y)Mf - F(x,y)\widehat{Mf}\right)dx^n.$$

Proof: Consider a sphere $U(y,\delta) \subset K$ with center at y and radius $\delta > 0$ then we have

$$\int_{\partial K} P\left(p(x,y)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)\right) + \frac{1}{y_n}\int_{\partial K} Q\left(q(x,y)n(x)f(x)d\sigma(x)\right)e_n$$

$$= \int_{\partial K\setminus\partial U(y,\delta)} P\left(p(x,y)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)\right)$$

$$+\frac{1}{y_n}\int_{\partial K\setminus\partial U(y,\delta)} Q\left(q(x,y)n(x)f(x)d\sigma(x)\right)e_n$$

$$+\int_{\partial U(y,\delta)} P\left(p(x,y)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)\right) + \frac{1}{y_n}\int_{\partial U(y,\delta)} Q\left(q(x,y)n(x)f(x)d\sigma(x)\right)e_n$$

$$= \int_{K\setminus U(y,\delta)} P\left(p(x,y)Mf\right)\frac{1}{x_n^{n-1}} + \frac{1}{y_n}Q\left(q(x,y)Mf\right)dx^n$$

$$+\int_{\partial U(y,\delta)} P\left(p(x,y)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)\right) + \frac{1}{y_n}\int_{\partial U(y,\delta)} Q\left(q(x,y)n(x)f(x)d\sigma(x)\right)e_n$$

When δ tends to 0 then

$$\int_{\partial U(y,\delta)} P\left(p(x,y)\frac{n(x)}{x_n^{n-2}}f(x)d\sigma(x)\right) + \frac{1}{y_n}\int_{\partial U(y,\delta)} Q(q(x,y)n(x)f(x)d\sigma(x))e_n$$

tends to $\frac{\omega_n}{(2y_n)^n}f(y)$. The result follows. Q.E.D.

Now let us note that

$$D_yE(x,y) = (n-2)\frac{(\hat{x}-y)(\overline{x}-\overline{y})}{\|\hat{x}-y\|^n\|x-y\|^n} \tag{2}$$

and

$$D_yF(x,y) = (n-2)\frac{(x-y)(\overline{\hat{x}}-\overline{y})}{\|\hat{x}-y\|^n|x-y|^n}. \tag{3}$$

Using these formulas we can deduce the following result.

Theorem 13 *Let $L \in C^1(\overline{K})$, then $I(y)$ is a hypermonogenic function on $R^n\setminus\overline{K}$ where*

$$I(y) = y_n^{n-2}\left(\int_K E(x,y)L(x)dx^n - \int_K F(x,y)\hat{L}(x)dx^n\right).$$

Proof: First we have

$$\frac{n-2}{y_n} Q'(I(y)) = \frac{n-2}{y_n} \left[\frac{\hat{I}(y) - I(y)}{2} e_n \right]'$$

$$= -\frac{n-2}{y_n} \left[\frac{\hat{I}(y) - I(y)}{2} \right]' e_n = -\frac{n-2}{2y_n} e_n \left[I(y) - \hat{I}(y) \right]$$

By (2) and (3) we have

$$M(I(y)) = DI(y) + \frac{n-2}{y_n} Q'(I(y))$$

$$= e_n (n-2) y_n^{n-3} \left[\int_K E(x,y) L(x) dx^n - \int_K F(x,y) \hat{L}(x) dx^n \right]$$

$$+ y_n^{n-2} \left[\int_K D_y E(x,y) L(x) dx^n - \int_K D_y F(x,y) L(\hat{x}) dx^n \right] - \frac{(n-2) e_n}{2y_n} \left[I(y) - \hat{I}(y) \right]$$

$$= \frac{(n-2) y_n^{n-3}}{2} \left[\int_K \frac{e_n (\overline{x} - \overline{y})}{\|x-y\|^n \|\hat{x} - y\|^{n-2}} L(x) dx^n \right.$$

$$+ \int_K \frac{2y_n (\hat{x} - y)(\overline{x} - \overline{y})}{\|x-y\|^n \|\hat{x} - y\|^n} L(x) dx^n$$

$$\left. - \int_K e_n \frac{(\overline{x} - \overline{\hat{y}})}{\|x-y\|^{n-2} \|\hat{x} - y\|^n} L(x) dx^n \right]$$

$$+ \frac{(n-2) y_n^{n-3}}{2} \left[\int_K \frac{-e_n (\overline{\hat{x}} - \overline{y})}{\|x-y\|^{n-2} \|\hat{x} - y\|^n} \hat{L}(x) dx^n \right.$$

$$- \int_K \frac{2y_n (x-y)(\overline{\hat{x}} - \overline{y})}{|x-y|^n \|\hat{x} - y\|^n} \hat{L}(x) dx^n$$

$$\left. + \int_K \frac{e_n (\overline{\hat{x}} - \overline{\hat{y}})}{\|x-y\|^n \|\hat{x} - y\|^{n-2}} \hat{L}(x) dx^n \right]$$

$$= \frac{(n-2) y_n^{n-3}}{2} \left[I_1 + I_2 \right].$$

Here

$$I_1 = \int_K \frac{e_n (\overline{x} - \overline{y}) \|\hat{x} - y\|^2 + 2y_n (\hat{x} - y)(\overline{x} - \overline{y}) - e_n (\overline{x} - \overline{\hat{y}}) \|x-y\|^2}{\|x-y\|^n \|\hat{x} - y\|^n} L(x) dx^n$$

and

$$e_n (\overline{x} - \overline{y}) \|\hat{x} - y\|^2 + 2y_n (\hat{x} - y)(\overline{x} - \overline{y}) - e_n (\overline{x} - \overline{\hat{y}}) \|x-y\|^2$$

$$= \Big(e_n \|\hat{x} - y\|^2 + 2y_n (\hat{x} - y) - (\hat{x} - y) e_n (x-y) \Big) (\overline{x} - \overline{y})$$

$$= (\hat{x} - y) \Big((\overline{\hat{x}} - \overline{y}) e_n - 2y_n e_n e_n - (\overline{\hat{x}} - \overline{\hat{y}}) e_n \Big) (\overline{x} - \overline{y})$$

$$= (\hat{x} - y)\Big((\overline{\hat{x}} - \overline{y}) - 2y_n e_n - (\overline{\hat{x}} - \overline{\hat{y}})\Big)e_n(\overline{x} - \overline{y}) = 0.$$

So $I_1 = 0$. Similarly

$$I_2 = \int_K \frac{-\|x-y\|^2 e_n(\overline{\hat{x}} - \overline{y}) - 2y_n(x-y)(\overline{\hat{x}} - \overline{y}) + e_n(\overline{\hat{x}} - \overline{\hat{y}})\|\hat{x} - y\|^2}{\|x-y\|^n\|\hat{x}-y\|^n}\hat{L}(x)dx^n$$

and

$$-(x-y)(\overline{x} - \overline{y})e_n(\overline{\hat{x}} - \overline{y}) - 2y_n(x-y)(\overline{\hat{x}} - \overline{y}) + (x-y)e_n\|\hat{x} - y\|^2$$

$$= (x-y)\Big(- (\overline{x} - \overline{y})(x - \hat{y}) + 2e_n y_n(x - \hat{y}) + (\overline{x} - \overline{\hat{y}})(x - \hat{y})\Big)e_n$$

$$= (x-y)\Big(- \overline{x} + \overline{y} + 2e_n y_n + \overline{x} - \overline{\hat{y}}\Big)(x - \hat{y})e_n = 0.$$

So $I_2 = 0$ and $MI = 0$. Q.E.D.

Theorem 14 *Let $F \in C^1(\overline{K})$, $y \in K$ then $M(I(y)) = F(y)$ where $I(y)$ is as defined in Theorem 11.*

Proof: First we show that $I(y)$ is a well defined function on K. To do this we only need to show that the integral

$$\int_{B(y,r)} E(x,y)F(x)dx^n - \int_{B(y,r)} F(x,y)\hat{L}(x)dx^n$$

is well defined for any ball $B(y,r) \subset K$. Since

$$\left\|\int_{B(y,r)} E(x,y)L(x)dx^n - \int_{B(y,r)} F(x,y)\hat{L}(x)dx^n\right\| \leq sup_{x\in B(y,r)}\|L\|C(n)\int_0^r ds$$

the integral is clearly finite. Then we can calculate $MI(y)$. For any fixed point $y \in K$ take any closed n-dimensional rectangle $R(y) \subset K$. Based on Theorem 13 we have

$$MI(y) = M\left[y_n^{n-2}\left(\int_K E(x,y)L(x)dx^n - \int_K F(x,y)\hat{L}(x)dx^n\right)\right]$$

$$= M\Bigg[y_n^{n-2}\Bigg(\int_{K\setminus\overline{R}} E(x,y)L(x)dx^n - \int_{K\setminus\overline{R}} F(x,y)\hat{L}(x)dx^n$$

$$+ \int_{\overline{R}} E(x,y)L(x)dx^n - \int_{\overline{R}} F(x,y)\hat{L}(x)dx^n\Bigg)\Bigg]$$

$$= M\left[y_n^{n-2}\left(\int_{\overline{R}} E(x,y)L(x)dx^n - \int_{\overline{R}} F(x,y)\hat{L}(x)dx^n\right)\right]$$

We consider

$$\lim_{h_j \to 0} \frac{1}{h_j} \left[\int_{R(y)} \left(E(x, y - h_j e_j) - E(x, y) \right) L(x) dx^n . \right.$$

$$\left. - \int_{R(y)} \left(F(x, y - h_j e_j) - F(x, y) \right) \hat{L}(x) dx^n \right]$$

$$= \lim_{h_j \to 0} \frac{1}{h_j} \left[\int_{R_1(y, h_1)} E(x, y - h_j e_j) L(x) dx^n \right.$$

$$+ \int_{R(y,h_j) \setminus R_3(y,h_j)} F(x, y) \left(L(x - h_j e_j) - L(x) \right) dx^n$$

$$- \int_{R_3(y,h_j)} E(x, y) L(x) dx^n - \int_{R_1(y,h_1)} F(x, y - h_j e_j) \hat{L}(x) dx^n$$

$$+ \int_{R(y,h_j) \setminus R_3(y,h_j)} F(x, y) \left(\hat{L}(x - h_j e_j) - \hat{L}(x) \right) dx^n$$

$$\left. - \int_{R_3(y,h_j)} F(x, y) \hat{L}(x) dx^n \right]$$

where $R_1(y, h_j)$ is the closed rectangle obtained from $R(y)$ by truncating $R(y)$ in the $-e_j^{'}$th direction a distance h_j from the face whose normal vector is $-e_j$, $R_2(y, h_j) = R(y) - R_1(y, h_j)$ while $R_3(y, h_j)$ is the closed rectangle obtained from $R_{(y)}$ by truncating $R(y)$ in the$e_j^{'}$th direction a distance h_j from the face whose normal vector is e_j. The width of both $R_1(y, h_j)$ and $R_3(y, h_j)$ in the $e_j^{'}$th direction is h_j. Consequently the previous limits evaluates to

$$\frac{1}{h_j} \left[\int_{Q_1(y,j)} \left(E(x, y) L(x) - F(x, y) \right) \hat{L}(x) dx^n \right.$$

$$- \int_{Q_2(y,j)} \left(E(x, y) L(x) - F(x, y) \right) \hat{L}(x) dx^n$$

$$\left. + \int_{R(y)} E(x, y) \frac{\partial L(x)}{\partial x_j} dx^n - \int_{R(y)} F(x, y) \frac{\partial \hat{L}(x)}{\partial x_j} dx^n \right] .$$

So $MI(y) = F(y)$ Q.E.D.

6 Möbius Transformations and the Hyperbolic Dirac-Hodge Operator and Hyperbolic Laplacian

We begin by establishing an invariance for the Cauchy Integral Formula under Möbius transformations. We begin by considering the case of Kelvin inversion

$In(x)=-x^{-1}$ for $x\neq 0$. Suppose that $f(y)$ is left hypermonogenic on a domain U in upper half space. For K a closed bounded subregion of U we have

$$f(y)=\frac{2^{n-2}y_n^{n-2}}{\omega_n}\int_{\partial K}\Big(\frac{(x-y)}{\|x-y\|^n\|x-\hat{y}\|^{n-2}}n(x)f(x)$$
$$+\frac{(\hat{x}-y)}{\|\hat{x}-y\|^n\|x-y\|^{n-2}}\hat{n}(x)\hat{f}(x)\Big)\,d\sigma(x)$$

for each y in the interior of K. If now $y=-v^{-1}$ and $x=-u^{-1}$ then $y_n=\frac{v_n}{\|v\|^2}$ and the integral formula becomes

$$f(-v^{-1})=\frac{2^{n-2}v_n^{n-2}}{\|v\|^{2n-4}\omega_n}\int_{\partial K^{-1}}\Big(v\|v\|^{2n-4}\frac{(u-v)}{\|u-v\|^n\|\hat{u}-v\|^{n-2}}n(u)u^{-1}f(-u^{-1})$$
$$+v\|v\|^{2n-4}\frac{(\hat{u}-v)}{\|\hat{u}-v\|^n\|u-v\|^{n-2}}\hat{n}(u)\hat{u}^{-1}\hat{f}(-u^{-1})\Big)\,d\sigma(u).$$

This expression simplifies to

$$v^{-1}f(-v^{-1})=\frac{2^{n-2}v_n^{n-2}}{\omega_n}\int_{\partial K^{-1}}\Big(\frac{(u-v)}{\|u-v\|^n\|u-\hat{v}\|^{n-2}}n(u)u^{-1}f(-u^{-1})$$
$$+\frac{(\hat{u}-v)}{\|\hat{u}-v\|^n\|u-v\|^{n-2}}\hat{n}(u)\hat{u}^{-1}\hat{f}(-u^{-1})\Big)\,d\sigma(u).$$

Similar results may be obtained for the other generators of the Möbius group. It follows that we have:

Theorem 15 *Suppose that $\psi(u)=(au+b)(cu+d)^{-1}$ is a Möbius transformation that leaves $R^{n,+}$ invariant. Suppose also that f is left hypermonogenic on a domain $U\subset R^{n,+}$ and K is a closed bounded subregion of U. Then*

$$J(\psi,v)f\big(\psi(u)\big)=\frac{2^{n-2}v_n^{n-2}}{\omega_n}\int_{\partial\psi^{-1}(K)}\Big(\frac{(u-v)}{\|u-v\|^n\|\hat{u}-v\|^{n-2}}n(u)J(\psi,u)f(\psi(u))$$
$$+\frac{(\hat{u}-v)}{\|\hat{u}-v\|^n\|u-v\|^{n-2}}\hat{n}(u)\hat{J}(\psi,u)\hat{f}(\psi(u))\Big)\,d\sigma(u)$$

where $J(\psi,u)=\frac{\widetilde{cu+d}}{\|cu+d\|^2}$.

Similarly one can take the function $I(y)$ set up in the statement of Theorem 14 and see that

$$I\big(\psi(v)\big)=\frac{v_n^{n-2}}{\omega_n\|v\|^{2n-4}}\int_{U^{-1}}\Big(v^{-1}\|v\|^{2n-2}\frac{(u-v)}{\|u-v\|^n\|\hat{u}-v\|^{n-2}}\frac{u}{\|u\|^4}L(\psi(u))$$
$$-v^{-1}\|v\|^{2n-2}\frac{(\hat{u}-v)}{\|\hat{u}-v\|^n\|u-v\|^{n-2}}\frac{\hat{u}}{\|u\|^4}\hat{L}(\psi(u))\Big)\,du^n.$$

This simplifies to

$$v^{-1}I\big(\psi(v)\big)=\frac{v_n^{n-2}}{\omega_n}\int_{U^{-1}}\left(E(u,v)\frac{u}{\|u\|^4}L\big(\psi(u)\big)-F(u,v)\frac{\hat{u}}{\|u\|^4}\hat{L}(u)\right)du^n.$$

If now we set $L(x)=M\phi(x)$ where ϕ has compact support in U and apply the operator M to the above equation we obtain

$$M\left(v^{-1}f\Big(\phi\big(\psi(v)\big)\Big)\right)=\frac{v}{\|v\|^4}M\Big(\phi\big(\psi(v)\big)\Big).$$

Again similar results may be obtained for other generators of the conformal group. It follows that we have:

Theorem 16 *Suppose that $\psi(u)=(au+b)(cu+d)^{-1}$ is a Möbius transformation that leaves $R^{n,+}$ invariant. Suppose also that ϕ is a C^1 function with support in U. Then*

$$M\Big(J(\psi,v)\phi\big(\psi(v)\big)\Big)=J'(\psi,v)M\Big(\phi\big(\psi(v)\big)\Big)$$

where $J'(\psi,v)=\frac{\widetilde{cv+d}}{\|cv+d\|^4}$.

This theorem provides us with intertwining operators for the differential operator M under actions of the conformal group.
Using the previous theorem and a standard partition of unity argument we have:

Proposition 5 *Suppose that $f:U\to Cl_n$ is a left hypermonogeinic function in the variable x and $x=\psi(v)=(av+b)(cv+d)^{-1}$ is a Möbius transformation preserving upper half space then the function $J(\psi,v)f(\psi(v))$ is left hypermonogenic in the variable v.*

This result was established in [24] using different techniques.
Let us now consider the constant hypermonogenic function $f(x)=-e_1$. By the previous results then under inversion we obtain the left hypermonogenic function $-v^{-1}e_1=v'^{-1}$. The function v'^{-1} is a direct analogue of the function $\frac{1}{z}$ from one complex variable. For each $k\in N$ the function $\frac{\partial^k v'^{-1}}{\partial v_1^k}=(-1)^k k!v'^{-k-1}$ is also left hypermonogenic on upper half space. Again by employing inversion it may now be observed that v'^k is left hypermonogenic for each $k\in N$. These functions are direct analogues of the functions z^k from one complex variable. That such functions are left hypermonogenic was first observed, using a different argument, in [24].
We will now proceed to find intertwining operators for the operators $\triangle_{R^{n,+}}$ and $\triangle'_{R^{n,+}}$ under Möbius transformations.
Using Proposition 2, Theorem 4 and (1) one may adapt the arguments used to establish Theorem 16 to determine that for any Möbius transformation ψ

preserving upper half space and any Cl_{n-1} valued C^2 function ϕ defined on a domain in upper half space

$$\triangle_{R^{n,+}}\Big(\phi\big(\psi(v)\big)\Big) = J_1(\psi, v)\triangle_{R^{n,+}}\Big(\phi\big(\psi(v)\big)\Big)$$

where $J_1(\psi, v) = \frac{1}{\|cv+d\|^4}$.
Similarly

$$\triangle'_{R^{n,+}}\Big(\phi\big(\psi(v)\big)\Big) = J_1(\psi, v)\triangle'_{R^{n,+}}\Big(\phi\big(\psi(v)\big)\Big).$$

It follows that if $\phi(x)$ is annihilated by $\triangle'_{R^{n,+}}$ then so is $\phi(\psi(v))$.

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